

SPECTRAL GEOMETRY OF ETA-EINSTEIN SASAKIAN MANIFOLDS

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ABSTRACT. We extend a result of Patodi for closed Riemannian manifolds to the context of closed contact manifolds by showing the condition that a manifold is an η -Einstein Sasakian manifold is spectrally determined. We also prove that the condition that a Sasakian space form has constant ϕ -sectional curvature c is spectrally determined.

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1. INTRODUCTION

The relationship between the spectrum of certain natural operators of Laplace type and the underlying geometry of a Riemannian manifold has been studied by many authors. Let (M, g) be a compact Riemannian manifold. Let Δ_p be the Laplace-Beltrami operator acting on the space of smooth p forms over a compact m -dimensional Riemannian manifold M . Patodi [15] established the following spectral characterization of space forms:

Theorem 1. *Let (M_i, g_i) be compact Riemannian manifolds without boundary. Assume that $\text{Spec}(\Delta_p, M_1) = \text{Spec}(\Delta_p, M_2)$ for $p = 0, 1, 2$. Then:*

- (1) *The manifold M_1 has constant scalar curvature c if and only if the manifold M_2 has constant scalar curvature c .*
- (2) *The manifold M_1 is Einstein if and only if the manifold M_2 is Einstein.*
- (3) *The manifold M_1 has constant sectional curvature c if and only if the manifold M_2 has constant sectional curvature c .*

Donnelly [6] and Gilkey and Sacks [8] extended Theorem 1 to the complex setting, and the present author extended Theorem 1 from the context of closed Riemannian manifolds to the context of compact Riemannian manifolds with boundaries [11]. See also related work [12].

A contact metric manifold M of dimension m with contact form η and associated metric g is called an η -Einstein manifold if the Ricci tensor ρ is given by

$$\rho = \alpha g + \beta \eta \otimes \eta \quad \text{for } \alpha, \beta \in C^\infty(M).$$

Note that α and β are constant if M is a η -Einstein Sasakian manifold of dimension ≥ 5 [1]; this fails if $\dim M = 3$ [9]. Also note that the η -Einstein tangent sphere bundle of a Riemannian manifold M of radius r equipped with the standard contact metric structure has constant functions α and β [4, 13] if $\dim M \geq 2$.

The study of η -Einstein metrics is related to the Sasakian Calabi problem [2]. Tanno [18] showed that Sasaki metric on the unit tangent sphere bundle of any sphere S^n is η -Einstein and D -homothetic deformation of this metric produces a homogeneous Einstein metric on $T_1 S^n$. We refer to [5, 10, 16, 21] for related work and some physical applications. In this paper, we shall extend our study in the Riemannian setting to the case of the contact geometry setting. The following is the main result of this paper:

Theorem 2. *Let $M_i = (M_i, \eta_i, g_i, \phi_i, \xi_i)$ be m_i -dimensional compact Sasakian manifolds without boundary with $m_i \geq 5$. Assume that $\text{Spec}(\Delta_p, M_1) = \text{Spec}(\Delta_p, M_2)$ for $p = 0, 1, 2$. Then:*

- (1) $m_1 = m_2$ and $\text{Vol}(M_1) = \text{Vol}(M_2)$.
- (2) M_1 has constant scalar curvature c if and only if the manifold M_2 has constant scalar curvature c .
- (3) M_1 is η -Einstein if and only if M_2 is η -Einstein.
- (4) M_1 is Sasakian space form with constant ϕ -sectional curvature c if and only if M_2 is Sasakian space form with constant ϕ -sectional curvature c .

The values $p = 0, 1, 2$ are not particularly special. They are chosen for illustrative purposes only - there are other values which could be chosen - see related work [17, 19] for example. If $f \in C^\infty(M)$, let

$$f[M] := \int_M f(x) \, \text{dvol}(x).$$

The crucial point is that under the hypotheses of either Theorem 1 or of Theorem 2 that

$$\{1[M], \tau[M], \tau^2[M], |\rho|^2[M], |R|^2[M]\}$$

are spectrally determined.

Here is a brief outline to the remainder of this paper. In Section 2, we review some facts concerning the Sasakian manifold. In Section 3, we review some previous results concerning the heat trace asymptotics. In Section 4, we complete the proof of Theorem 2.

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2. SASAKIAN MANIFOLDS

All manifolds in the present paper are assumed to be connected and of class C^∞ . We prepare some fundamental material about Sasakian manifold. We refer to [1] for further details. A $(2n+1)$ -dimensional manifold M^{2n+1} is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η , we have a unique vector field ξ , the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X . It is well-known that there exists a Riemannian metric g and a $(1, 1)$ -tensor field ϕ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi \quad (1)$$

where X and Y are vector fields on M . From (1), it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2)$$

A Riemannian manifold M equipped with structure tensors (η, g, ϕ, ξ) satisfying (1) is said to be a *contact metric manifold* (or *contact Riemannian manifold*) and is denoted by $M = (M, \eta, g, \phi, \xi)$.

A normal contact metric manifold is called a *Sasakian manifold*. Equivalently, an almost contact metric manifold $M = (M, \eta, g, \phi, \xi)$ is a Sasakian manifold if and only if the following condition holds [1]:

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X. \quad (3)$$

On the other hand, a contact metric manifold is called a *K-contact manifold* if the characteristic vector field ξ is a Killing vector field. It is well-known that a Sasakian

manifold is necessarily a K -contact manifold. We also have the following formulas for a Sasakian manifold [1].

$$\nabla_X \xi = -\phi X, \quad (4)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (5)$$

$$\rho(\xi, \xi) = 2n, \quad (6)$$

where R and ρ are the curvature tensor and Ricci tensor of M , respectively.

Definition 1. A contact metric manifold $M = (M, \eta, g, \phi, \xi)$ is said to be η -Einstein if the Ricci tensor ρ of M is of the form

$$\rho = \alpha g + \beta \eta \otimes \eta$$

for smooth functions α and β on M .

On the other hand, it is known that any 3-dimensional Sasakian manifold is η -Einstein. We may easily check that, for an η -Einstein Sasakian manifold, the functions α and β are both constant if $\dim M \geq 5$ [1]. A Sasakian manifold M is called a *Sasakian space form* if M has constant ϕ -sectional curvature. It is known that the curvature tensor [20] of a $2n + 1 \geq 5$ -dimensional Sasakian space form with constant ϕ -sectional curvature is given by

$$\begin{aligned} R(X, Y, Z, W) &= g(R(X, Y)Z, W) \\ &= \frac{c+3}{4}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &\quad + \frac{c-1}{4}\{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) + 2g(X, \phi Y)g(\phi Z, W)\} \\ &\quad + \frac{c-1}{4}\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) + g(X, Z)\eta(Y)\eta(W) \\ &\quad - g(Y, Z)\eta(X)\eta(W)\}. \end{aligned} \quad (7)$$

for any vector fields X, Y, Z, W on M . Let M be a $(2n + 1)$ -dimensional Sasakian space form. We set $m = 2n + 1$. Then from (7), we see that the Ricci tensor ρ of M is given by

$$\rho = \frac{1}{4}\{(m+1)c + 3m - 5\}g - \frac{m+1}{4}(c-1)\eta \otimes \eta, \quad (8)$$

and hence, M is an η -Einstein manifold.

We now define the tensor fields $S_{\alpha, \beta}$ and T_c of M respectively by

$$S_{\alpha, \beta}(X, Y) = \rho(X, Y) - (\alpha g(X, Y) + \beta \eta(X)\eta(Y)), \quad \text{and} \quad (9)$$

$$\begin{aligned} T_c(X, Y, Z, W) &= R(X, Y, Z, W) - \left\{ \frac{c+3}{4}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \right. \\ &\quad + \frac{c-1}{4}\{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) + 2g(X, \phi Y)g(\phi Z, W)\} \\ &\quad + \frac{c-1}{4}\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) + g(X, Z)\eta(Y)\eta(W) \\ &\quad \left. - g(Y, Z)\eta(X)\eta(W)\} \right\}. \end{aligned} \quad (10)$$

for vector fields X, Y, Z, W on M , where α, β are some smooth functions on M and c is a constant.

Let $\{e_i\}$ be an orthonormal basis of $T_p M$ at any point $p \in M$. In the sequel, we shall adopt the following notational convention:

$$\begin{aligned} R_{ijkl} &= g(R(e_i, e_j)e_k, e_l), \\ \rho_{ij} &= \rho(e_i, e_j), \quad \phi_{ij} = g(\phi e_i, e_j), \\ \nabla_i \phi_{jk} &= g((\nabla_{e_i} \phi)e_j, e_k), \quad \nabla_i \eta_j = g((\nabla_{e_i} \xi), e_j) \end{aligned} \quad (11)$$

and so on, where the Latin indices run over the range $1, 2, \dots, m = 2n + 1$. We adopt the Einstein summation convention for the repeated indices. From (3) to (6), we may rewrite as follows:

$$\begin{aligned} \nabla_i \phi_{jk} &= g_{ij} \eta_k - \eta_j g_{ik}, \\ \nabla_i \eta_j &= -\phi_{ij}, \\ R_{ijkl} \eta_k &= \eta_j g_{il} - \eta_i g_{jl}, \\ \rho_{ij} \eta_i \eta_j &= 2n. \end{aligned} \quad (12)$$

From the definition of the tensor field $S_{\alpha, \beta}$ and (12), by direct calculation, we have

$$|S_{\alpha, \beta}|^2 = |\rho|^2 - 2\alpha\tau + \gamma, \quad (13)$$

where $\gamma = m\alpha^2 + 2\alpha\beta + \beta^2 - 2(m-1)\beta$. Further, we see that M is η -Einstein with the coefficient functions α and β in the defining equation if and only if $S_{\alpha, \beta} = 0$ and $\alpha = \frac{\tau}{m-1} - 1$, $\beta = m - \frac{\tau}{m-1}$ hold. In this case, we note that α , β and τ are all constant if $\dim M \geq 5$.

Next, we prepare the following Lemma to calculate the square norm $|T_c|^2$ of the tensor field T_c on M .

Lemma 3. *On Sasakian manifold, we have*

$$R_{ijkl} \{\phi_{ki} \phi_{jl} - \phi_{kj} \phi_{il} + 2\phi_{ji} \phi_{kl}\} = 6\tau - 6(m-1)^2.$$

Proof First, we get

$$R_{ijkl} \phi_{ki} \phi_{jl} = \frac{1}{2}(R_{ijkl} - R_{kji l}) \phi_{ki} \phi_{jl} = \frac{1}{2}(R_{ijkl} + R_{jkil}) \phi_{ki} \phi_{jl} = -\frac{1}{2} R_{kijl} \phi_{ki} \phi_{jl}. \quad (14)$$

Similarly, we obtain

$$\begin{aligned} -R_{ijkl} \phi_{kj} \phi_{il} &= -\frac{1}{2} R_{jkil} \phi_{jk} \phi_{il}, \\ R_{ijkl} \phi_{ji} \phi_{kl} &= -R_{ijkl} \phi_{ij} \phi_{kl}. \end{aligned} \quad (15)$$

From (14) and (15), we have

$$R_{ijkl} \{\phi_{ki} \phi_{jl} - \phi_{kj} \phi_{il} + 2\phi_{ji} \phi_{kl}\} = -3R_{ijkl} \phi_{ij} \phi_{kl}. \quad (16)$$

On the other hand, from (12), we get

$$\nabla_l \nabla_i \phi_{jk} = g_{ij} \nabla_l \eta_k - g_{ik} \nabla_l \eta_j = -g_{ij} \phi_{lk} + g_{ik} \phi_{lj},$$

and hence

$$\nabla_l \nabla_i \phi_{jk} - \nabla_i \nabla_l \phi_{jk} = -g_{ij} \phi_{lk} + g_{ik} \phi_{lj} + g_{lj} \phi_{ik} - g_{lk} \phi_{ij}. \quad (17)$$

Applying the Ricci identity to (17), and then taking sum by setting $i = k$ in the resulting equality, we get

$$-R_{l i j a} \phi_{ai} - \rho_{la} \phi_{ja} = (m-2) \phi_{lj}. \quad (18)$$

Transvecting ϕ_{lj} to (18), and taking account of (12), we have

$$-R_{l i j a} \phi_{ai} \phi_{lj} - \rho_{la} \phi_{ja} \phi_{lj} = (m-2) \phi_{lj} \phi_{lj} = (m-1)(m-2), \quad (19)$$

and hence

$$-R_{ilaj} \phi_{ai} \phi_{jl} = -\rho_{la} (g_{la} - \eta_l \eta_a) + (m-1)(m-2) = -\tau + (m-1)^2. \quad (20)$$

Thus, from (14) and (20), we have

$$\frac{1}{2}R_{jkil}\phi_{jk}\phi_{il} = -\tau + (m-1)^2. \quad (21)$$

Therefore, from (16) and (21), we have

$$R_{ijkl}\{\phi_{ki}\phi_{jl} - \phi_{kj}\phi_{il} + 2\phi_{ji}\phi_{kl}\} = 6\tau - 6(m-1)^2. \quad (22)$$

This completes the proof of Lemma 3. \square

3. HEAT TRACE ASYMPTOTICS

Let M be a compact Riemannian manifold of real dimension m without boundary, and let D be a operator of Laplace type on the space of smooth sections to a smooth vector bundle over M . Let e^{-tD} be the fundamental solution of the heat equation. This operator is of trace class and as $t \downarrow 0$ there is a complete asymptotic expansion with locally computable coefficients in the form:

$$\mathrm{Tr}_{L^2} e^{-tD} \sim \sum_{n \geq 0} t^{(n-m)/2} a_n(D).$$

To study the heat trace coefficients $a_n(D)$, we introduce a bit of additional notation. There is a canonically defined connection $\nabla = \nabla(D)$ and a canonically defined endomorphism $E = E(D)$ so that

$$D = -(\mathrm{Tr}(\nabla^2) + E).$$

Let indices i, j, k range from 1 to m and index a local orthonormal frame $\{e_1, \dots, e_m\}$ for TM . Let Ω be the curvature of ∇ , let $\tau := R_{ijji}$ be the scalar curvature, let $\rho_{ij} := R_{ikkk}$ be the Ricci tensor. Let $'$ denote multiple covariant differentiation. We refer to [3] for the proof of the following result:

Theorem 4. *Let D be an operator of Laplace type on the space of sections $C^\infty(V)$ to a vector bundle V over a compact manifold M . Let I be the identity endomorphism of V . We have:*

- (1) $a_0(D) = (4\pi)^{-m/2} \int_M \mathrm{Tr}\{I\}.$
- (2) $a_2(D) = (4\pi)^{-m/2} \frac{1}{6} \int_M \mathrm{Tr}\{6E + \tau I\}.$
- (3) $a_4(D) = (4\pi)^{-m/2} \frac{1}{360} \int_M \mathrm{Tr}\{60E_{;kk} + 60\tau E + 180E^2 + 30\Omega^2 + (12\tau_{;kk} + 5\tau^2 - 2|\rho|^2 + 2|R|^2)I\}.$

Theorem 4 plays an important role in the proof of Theorem 2. We refer to [7] for further details.

4. PROOF OF THEOREM 2

Let $M = (M, \eta, g, \phi, \xi)$ be a $2n+1 \geq 5$ -dimensional compact Sasakian manifold without boundary, and set $m = 2n+1$. From Theorem 4, for $D = \Delta_p$ ($p = 0, 1, 2$), we have

$$\mathrm{Tr}_{L^2}(e^{-t\Delta_0}) = (4\pi t)^{-m/2} \{\mathrm{Vol}(M) + O(t)\} \quad \text{and also} \quad (23)$$

$$a_2(\Delta_0, M) = \frac{1}{6} (4\pi)^{-m/2} \int_M \tau,$$

$$a_2(\Delta_1, M) = \frac{1}{6} (4\pi)^{-m/2} \int_M (m-6)\tau, \quad (24)$$

The work of Patodi [15] shows that there exist universal constants so:

$$a_4(\Delta_p, M) = (4\pi)^{-m/2} \int_M \{c_{m,p}^1 \tau^2 + c_{m,p}^2 |\rho|^2 + c_{m,p}^3 |R|^2 + c_{m,p}^4 \tau_{;ii}\}. \quad (25)$$

$p = 0, 1, 2$.

Now, we shall prove Theorem 2. Let $M_i = (M_i, \eta_i, g_i, \phi_i, \xi_i)$ be m_i -dimensional compact Sasakian manifolds without boundary of $m_i \geq 5$ ($i = 1, 2$). Assume that $\text{Spec}(\Delta_p, M_1) = \text{Spec}(\Delta_p, M_2)$ for $p = 0, 1, 2$. We denote by R_i , ρ_i and τ_i the curvature tensor, the Ricci tensor and the scalar curvature of M_i ($i = 1, 2$), respectively. Then, from $a_0(\Delta_0, M_i)$ in Theorem 4 (1), we have

$$m_1 = m_2 \quad \text{and} \quad \text{Vol}(M_1) = \text{Vol}(M_2). \quad (26)$$

We then establish assertion (2) by computing:

$$\begin{aligned} \tau_1 &= (4\pi)^{m/2} \text{Vol}(M_1)^{-1} \{ma_2(\Delta_0, M_1) - a_2(\Delta_1, M_1)\} \\ &= (4\pi)^{m/2} \text{Vol}(M_2)^{-1} \{ma_2(\Delta_0, M_2) - a_2(\Delta_1, M_2)\} \\ &= \tau_2. \end{aligned} \quad (27)$$

The assertion (2) is nothing but a special case of the Theorem 1 (1).

Next, suppose that M_1 is an η -Einstein manifold with the coefficient functions α_1 and β_1 in the defining equation. Since $m_1 \geq 5$, it follows that α_1 and β_1 are constant and hence, the scalar curvature τ_1 of M_1 is also constant given by $\tau_1 = m\alpha_1 + \beta_1$. Thus, from assertion (2), it follows that the scalar curvature τ_2 of M_2 is also constant and $\tau_1 = \tau_2$. Since $\text{Vol}(M_1) = \text{Vol}(M_2)$, the integrals of τ^2 are equal. Since $\tau_{;ii} = 0$, from (25), we have

$$\int_{M_1} (c_{m,p}^2 |\rho_1|^2 + c_{m,p}^3 |R_1|^2) = \int_{M_2} (c_{m,p}^2 |\rho_2|^2 + c_{m,p}^3 |R_2|^2) \quad (28)$$

for $p = 1, 2$; these two equations are independent [15]. Consequently

$$\int_{M_1} |\rho_1|^2 = \int_{M_2} |\rho_2|^2 \quad \text{and} \quad \int_{M_1} |R_1|^2 = \int_{M_2} |R_2|^2. \quad (29)$$

Thus, from (13) we have

$$0 = \int_{M_1} |S_{\alpha_1, \beta_1}^1|^2 = \int_{M_1} |\rho_1|^2 - 2\alpha_1\tau_1 + \gamma_1 = \int_{M_2} |\rho_2|^2 - 2\alpha_1\tau_1 + \gamma_1, \quad (30)$$

where $\gamma_1 = m\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 - 2(m-1)\beta_1$. Here, we may note that

$$\begin{aligned} \alpha_1 &= \frac{\tau_1}{m-1} - 1 = \frac{\tau_2}{m-1} - 1, \\ \beta_1 &= m - \frac{\tau_1}{m-1} = m - \frac{\tau_2}{m-1}. \end{aligned} \quad (31)$$

We here set

$$S_{\alpha, \beta}^2 = \rho_2 - (\alpha_2 g_2 + \beta_2 \eta_2 \otimes \eta_2), \quad \text{where} \quad \alpha_2 = \frac{\tau_2}{m-1} - 1, \beta_2 = m - \frac{\tau_2}{m-1}. \quad (32)$$

Then, we have

$$\int_{M_2} |S_{\alpha_2, \beta_2}^2|^2 = \int_{M_2} |\rho_2|^2 - 2\alpha_2\tau_2 + \gamma_2, \quad (33)$$

where $\gamma_2 = m\alpha_2^2 + 2\alpha_2\beta_2 + \beta_2^2 - 2(m-1)\beta_2$.

From (31) and (32), we get

$$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_2 \quad \text{and hence} \quad \gamma_1 = \gamma_2. \quad (34)$$

Therefore, from (30), (33) and (34), we have $0 = \int_{M_2} |S_{\alpha_2, \beta_2}^2|^2$, and therefore, M_2 is an η -Einstein manifold with the same coefficients in the defining equation. This completes the proof of Theorem 2 (3).

Lastly, suppose that M_1 is a Sasakian space form with constant ϕ -sectional curvature c . Then, from (8), we see that M_1 is an η -Einstein manifold with constant

coefficients $\alpha_1 = \frac{1}{4}\{(m+1)c + 3m - 5\}$ and $\beta_1 = -\frac{m+1}{4}(c-1)$ in the defining equation. Thus, it follows that the scalar curvature τ_1 is given by

$$\tau_1 = \frac{m-1}{4}\{(m+1)c + 3m - 1\}.$$

Thus, by the assertion (3) and hypothesis that $\text{Spec}(\Delta_p, M_1) = \text{Spec}(\Delta_p, M_2)$ ($p=0, 1, 2$), we see that M_2 is an η -Einstein manifold with the constant coefficients α_2 and β_2 in the defining equation such that $\alpha_2 = \alpha_1$, $\beta_2 = \beta_1$, and hence $\tau_2 = \tau_1$. We denote by T_c^2 the tensor field defined by (10) of the Sasakian manifold M_2 . We set

$$(T_c^2)_{ijkl} := R_{ijkl} - K_{ijkl}, \quad (35)$$

where

$$\begin{aligned} K_{ijkl} &= \frac{c+3}{4}\{g_{jk}g_{il} - g_{ik}g_{jl}\} + \frac{c-1}{4}\{\phi_{ki}\phi_{jl} - \phi_{kj}\phi_{il} + 2\phi_{ji}\phi_{kl}\} \\ &\quad + \frac{c-1}{4}\{\eta_i\eta_k g_{jl} - \eta_j\eta_k g_{il} + g_{ik}\eta_j\eta_l - g_{jk}\eta_i\eta_l\}. \end{aligned}$$

Then, by direct calculation, we have

$$\begin{aligned} |K|^2 &= \frac{(c+3)^2}{16}\{g_{jk}g_{il} - g_{ik}g_{jl}\}^2 + \frac{(c-1)^2}{16}(\phi_{ki}\phi_{jl} - \phi_{kj}\phi_{il} + 2\phi_{ji}\phi_{kl})^2 \\ &\quad + \frac{(c-1)^2}{16}(\eta_i\eta_k g_{jl} - \eta_j\eta_k g_{il} + g_{ik}\eta_j\eta_l - g_{jk}\eta_i\eta_l)^2 \\ &\quad + \frac{(c-1)(c+3)}{8}(g_{jk}g_{il} - g_{ik}g_{jl})(\phi_{ki}\phi_{jl} - \phi_{kj}\phi_{il} + 2\phi_{ji}\phi_{kl}) \\ &\quad + \frac{(c-1)(c+3)}{8}(g_{jk}g_{il} - g_{ik}g_{jl})(\eta_i\eta_k g_{jl} - \eta_j\eta_k g_{il} + g_{ik}\eta_j\eta_l - g_{jk}\eta_i\eta_l) \\ &\quad + 0 \\ &= \frac{m-1}{2}\{(m+1)c^2 + 3m - 1\}. \end{aligned} \quad (36)$$

Next, by taking account of Lemma 3, we get

$$\begin{aligned} R_{ijkl}K_{ijkl} &= \frac{c+3}{4}R_{ijkl}(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}) \\ &\quad + \frac{c-1}{4}R_{ijkl}(\phi_{ki}\phi_{jl} - \phi_{kj}\phi_{il} + 2\phi_{ji}\phi_{kl}) \\ &\quad + \frac{c-1}{4}R_{ijkl}(\eta_i\eta_k\delta_{jl} - \eta_j\eta_k\delta_{il} + \delta_{ik}\eta_j\eta_l - \delta_{jk}\eta_i\eta_l) \\ &= 2c\tau - \frac{1}{2}(m-1)(3m-1)(c-1). \end{aligned} \quad (37)$$

Then, from (35), (36) and (37), we have

$$|T_c^2|^2 = |R_2|^2 - 4c\tau_2 + d, \quad (38)$$

where $d := \frac{m-1}{2}(m+1)c^2 + (m-1)(3m-1)c - \frac{1}{2}(m-1)(3m-1)$. On the other hand, since M_1 is m -dimensional Sasakian space form with constant ϕ -sectional curvature c , we have

$$|R_1|^2 = \frac{m-1}{2}\{(m+1)c^2 + 3m - 1\},$$

and further

$$0 = |T_c^1|^2 = |R_1|^2 - 4c\tau_1 + d. \quad (39)$$

Then we use (29), (38), taking account of (39) and $\tau_1 = \tau_2$, we have

$$0 = \int_{M_1} |T_c^1|^2 = \int_{M_1} |R_1|^2 - 4c\tau_1 + d = \int_{M_2} |R_2|^2 - 4c\tau_2 + d = \int_{M_2} |T_c^2|^2,$$

and hence, $T_c^2 = 0$ on M_2 . Therefore, we see that M_2 is also an m -dimensional Sasakian space form with constant ϕ -sectional curvature c . This completes the proof of Theorem 2 (4). \square

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